

ON THE TRANSVERSE SCALAR CURVATURE OF A COMPACT SASAKI MANIFOLD

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ABSTRACT. We show that the *standard picture* regarding the notion of stability of constant scalar curvature metrics in Kähler geometry described by S.K. Donaldson [8, 9], which involves the geometry of infinite-dimensional groups and spaces, can be applied to the constant scalar curvature metrics in Sasaki geometry with only few modification. We prove that the space of Sasaki metrics is an infinite dimensional symmetric space and that the transverse scalar curvature of a Sasaki metric is a moment map of the strict contactomorphism group.

1. INTRODUCTION

Sasaki geometry, in particular Sasaki-Einstein manifolds, has been extensively studied. Readers are referred to recent monograph Boyer-Galicki [1], and recent survey paper Sparks [30] and the references in for history, background and recent progress of Sasaki geometry and Sasaki-Einstein manifolds.

Sasaki geometry is often described as an odd-dimensional analogue of Kähler geometry. Some important results in Kähler geometry have now a counterpart in the Sasaki context. For example, Calabi's extremal problem in Kähler geometry has been extended to the Sasaki manifolds and some important results have been obtained [2, 3, 16]. In this note we shall discuss the notion of stability of constant scalar curvature metrics in Sasaki context. One can see that the picture regarding the stability of constant scalar curvature described by S.K. Donaldson [8, 9] can be carried over to the Sasaki setting with only slight modification.

It is well known in Kähler geometry that there are obstructions to the existence of extremal metrics in general [26, 21, 15, 5]; there is now a very beautiful picture regarding the existence of constant scalar curvature metrics on algebraic varieties, which is conjectured to be equivalent to certain notion of stability in algebraic geometry [33, 32, 11]; this picture can also be generalized to extremal metrics [24, 31]. There has also been important progress to understand obstructions of canonical metrics in Sasaki geometry [17, 16, 2]. In particular, Gauntlett-Martelli-Sparks-Yau [17] has proved the *Lichnerowicz obstruction* and the *Bishop obstruction* to existence of Sasaki Einstein metrics. Their results provide new obstructions to existence of Kähler-Einstein metrics on Kähler orbifolds. Recently Ross-Thomas [28] studied stability of Kähler orbifolds and constant scalar curvature metrics on Kähler orbifolds.

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In [8, 9], Donaldson suggested that the stability of Kähler-Einstein metrics [33, 32] fits into a general framework, involving the geometry of infinite dimensional groups and spaces; from this point of view, the notion of stability can be extended to, for example, the metrics of constant scalar curvature. One of the key observations relating canonical metrics in Kähler geometry with the geometry of infinite dimensional groups and spaces is that the scalar curvature of a Kähler manifold (V, ω) can be interpreted as a moment map of symplectomorphism group of (V, ω) [14, 8]. We shall show similar pictures described in [8, 9] can be extended to Sasaki geometry with only slight modification. Our discussion follows closely the discussion by S.K. Donaldson [8, 9] in (almost) Kähler setting. While the picture described by Donaldson stimulates tremendous work in Kähler geometry; it might still be worth to show that the similar picture holds for the Sasaki case, even though there is no new idea beyond [8, 9] and there is only few modification to prove the results.

In Section 3 we consider the space of Sasaki metrics studied in [18, 19], which is the analogue of the space of Kähler metrics as in [23, 29, 9]. Guan-Zhang [18, 19] studied the geometry of the space of Sasaki metrics, in particular the geodesic equations. They proved, among others, the regularity results of the Dirichlet problem of the geodesic equation with applications in Sasaki geometry, parallel the results of Chen [7] and Calabi-Chen [6] in the space of Kähler metrics. As a consequence of their results, the space of Sasaki metrics is a non-positively curved metric space. We observe that the space of Sasaki metrics is an infinite dimensional symmetric space as in the Kähler setting [23, 29, 9]. Moreover, \mathcal{H} can be viewed as the dual space of the strict contactomorphism group \mathcal{G} .

In Section 4 we show that the transverse scalar curvature of a Sasaki metric (K-contact metric) can be interpreted as a moment map of strict contactomorphism group; hence the “standard picture” described in [8, 9] can be extended to Sasaki geometry. Thus it is not a surprise that the existence of canonical metrics in Sasaki geometry should also be relevant to the notion of stability, see [28] for recent progress. Following [9], we shall present a problem regarding nonexistence of constant transverse scalar curvature metrics, using the notion of geodesic rays and the K -energy in Sasaki geometry [22, 19].

2. SASAKI GEOMETRY, THE CONTACT STRUCTURE AND THE TRANSVERSE KÄHLER STRUCTURE

A Sasaki manifold involves many interesting structures, including its underlying contact structure and the transverse Kähler structure. In this section we recall these relevant structures on Sasaki manifolds, or more generally, K-contact manifolds.

Let (M, η) be a compact contact manifold of dimension $2n + 1$ with a contact 1-form η such that $\eta \wedge (d\eta)^n$ is nowhere vanishing. The Reeb vector field ξ is defined uniquely by

$$\eta(\xi) = 1, \iota_\xi d\eta = 0.$$

The 1-dimensional foliation generated by ξ is called the *Reeb foliation*. The contact form η defines a vector sub-bundle D of the tangent bundle TM such that $D = \ker(\eta)$ and $TM = D \oplus L\xi$, where $L\xi$ is the trivial line bundle generalized by ξ . A contactomorphism $f : M \rightarrow M$ is a diffeomorphism which satisfies $f^*\eta = \exp(F)\eta$ for some function $F : M \rightarrow \mathbb{R}$. The group \mathfrak{G} of contactomorphisms leaves $D = \ker(\eta)$ invariant. With the 1-form η fixed we can also define the group \mathcal{G} of strict

contactomorphisms by the condition $f^*\eta = \eta$; it is clear that \mathcal{G} is a subgroup of \mathfrak{G} . We denote the Lie algebra of \mathfrak{G} and \mathcal{G} by \mathfrak{h} and \mathfrak{g} respectively. The Lie algebra can be characterized by

$$\begin{aligned}\mathfrak{h} &= \{X \in \Gamma(TM), L_X\eta = F\eta \text{ for some } F \in C^\infty(M)\}, \\ \mathfrak{g} &= \{X \in \Gamma(TM), L_X\eta = 0\}.\end{aligned}$$

A vector field X is called contact if $L_X\eta = F\eta$ and it is called strict contact if $L_X\eta = 0$. There is a Lie algebra isomorphism between the space $C^\infty(M)$ of functions on M and the Lie algebra \mathfrak{h} , see [27, 1] for the details. For every function $H : M \rightarrow \mathbb{R}$, there exists a unique contact vector field $X = X_H : M \rightarrow TM$ which satisfies

$$H = \eta(X), \iota_X d\eta = dH(\xi)\eta - dH.$$

The Poisson bracket is then defined by

$$\{F, H\} = \eta([X_F, X_H]).$$

There is also a natural L^2 inner product on $C^\infty(M)$:

$$\langle f, h \rangle = \int_M f h d\mu,$$

where $d\mu = (2^n n!)^{-1} \eta \wedge (d\eta)^n$ is a volume form determined by η . However this L^2 product is not \mathfrak{G} -invariant. Instead we consider the subgroup \mathcal{G} and the space $C_B^\infty(M)$ of basic functions on M , where

$$C_B^\infty(M) = \{\phi \in C^\infty(M) : d\phi(\xi) = 0\}.$$

Then $C_B^\infty(M)$ is a sub-algebra of $C^\infty(M)$ and it is isomorphic to the Lie algebra of \mathfrak{g} (for example, see [1] Section 6.1). The L^2 inner product, restricted on $C_B^\infty(M)$, is \mathcal{G} -invariant. We can write $C_B^\infty(M) = C_{B,0}^\infty(M) \oplus \mathbb{R}$, where $C_{B,0}^\infty(M)$ is the space of functions of integral zero, the L^2 orthogonal complement of the constants. Now the groups \mathcal{G} has a bi-invariant metric defined by L^2 inner product on its Lie algebra, so it furnishes an example of infinite dimensional symmetric space. When M is a Sasaki manifold, we shall show in next section that this space has a *negatively curved dual*.

First we recall the *K-contact structure*. Let (M^{2n+1}, η, ξ) be a compact contact manifold.

Definition 2.1. A $(1,1)$ -tensor field $\Phi : TM \rightarrow TM$ is called an *almost contact-complex structure* if

$$\Phi\xi = 0, \Phi^2 = -id + \xi \otimes \eta.$$

It is called an *K-contact-complex structure* if in addition, $L_\xi\Phi = 0$. An *almost contact-complex structure* is compatible with η if

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y); d\eta(X, \Phi X) > 0, X \in \mathcal{D}, X \neq 0.$$

If Φ is compatible with η , (M, η, Φ) defines a Riemannian metric

$$g(X, Y) = \frac{1}{2} d\eta(X, \Phi Y) + \eta(X)\eta(Y),$$

and (M, η, ξ, Φ, g) is called a contact metric structure. This metric structure is called a K-contact metric structure if $L_\xi\Phi = 0$, which corresponds to an almost

Kähler metric on a symplectic manifold. A Sasaki structure can then be defined as a K-contact metric structure (M, η, ξ, Φ, g) where Φ satisfies an integrable condition

$$\nabla \Phi(X, Y) = g(\xi, Y)X - g(X, Y)\xi.$$

This definition is equivalent to the following characterization: a Riemannian metric (M, η, ξ, Φ, g) is called Sasaki if the cone metric $(C(M) = M \times \mathbb{R}_+, dr^2 + r^2g)$ is Kähler. The complex structure J on $C(M)$ can be defined as

$$J\left(r\frac{\partial}{\partial r}\right) = \xi, JY = \Phi Y - \eta(Y)r\frac{\partial}{\partial r}, Y \in \Gamma(TM).$$

For a Sasaki structure, there are two relevant Kähler structures. One is the Kähler cone structure on $C(M)$ and the other is the transverse Kähler structure for the Reeb foliation. For our purpose, we shall also consider the *transverse almost Kähler* structure for a K-contact metric structure (M, η, ξ, Φ, g) . The discussion is similar as in the transverse Kähler structure [1, 16]. We shall describe the transverse almost Kähler structure both globally and locally.

Let $\{U_\alpha\}_{\alpha \in I} \subset \mathbb{R}^{2n+1}$ be an open covering of M and let $\pi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{R}^{2n}$ be submersions such that $d\pi_\alpha(\xi) = 0$ and when $U_\alpha \cap U_\beta \neq \emptyset$

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \rightarrow \pi_\alpha(U_\alpha \cap U_\beta)$$

is diffeomorphic. On each V_α , we can define an almost Kähler metric as follows.

Let $(x^1, x^2, \dots, x^{2n})$ be a local coordinate on V_α . We can pull this back on U_α and still write them as x^1, x^2, \dots, x^{2n} . Let x^0 be the coordinate along the leaves with $\xi = \frac{\partial}{\partial x^0}$. Then $(x^0, x^1, \dots, x^{2n})$ forms a local coordinate on U_α . Suppose we can write $\eta = dx^0 - a_i dx^i$ locally for some functions a_i , $1 \leq i \leq 2n$. Since $\iota_\xi d\eta = 0$, it is clear that $da_i(\xi) = 0$ and a_i is a function of x^1, \dots, x^{2n} . We can then define

$$\omega_\alpha := \frac{1}{2} \left(\frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} \right) dx^i \wedge dx^j = \omega_{ij} dx^i \wedge dx^j.$$

It is clear that ω_α coincides with $\frac{1}{2}d\eta$ on $\{x^0 = \text{const}\} \subset U_\alpha$. We also get that \mathcal{D} is spanned by the vectors of the form

$$e_i = \frac{\partial}{\partial x^i} + a_i \frac{\partial}{\partial x^0}, i = 1, 2, \dots, 2n.$$

For any $p \in U_\alpha$ there is an isomorphism induced by π_α

$$d\pi_\alpha : \mathcal{D}_p \rightarrow T_{\pi_\alpha(p)} V_\alpha; d\pi_\alpha(e_i) = \frac{\partial}{\partial x^i}.$$

An almost contact-complex structure K can be written as

$$\Phi e_i = \Phi_i^j e_j, \Phi_i^j \Phi_j^k = -\delta_i^k.$$

A simple computation shows that the condition $\Phi_\xi K = 0$ implies that

$$\frac{\partial \Phi_i^j}{\partial x^0} = 0.$$

Hence we can define an almost complex structure on V_α such that

$$\Phi_\alpha \frac{\partial}{\partial x_i} = \Phi_i^j \frac{\partial}{\partial x_j}.$$

It is clear that Φ_α is compatible with ω_α ; thus we can define an almost Kähler metric g_α^T on V_α . By this construction,

$$\pi_\alpha \circ \pi_\beta^{-1} : \pi_\beta(U_\alpha \cap U_\beta) \rightarrow \pi_\alpha(U_\alpha \cap U_\beta)$$

gives an isometry of almost Kähler manifolds. The collection of almost Kähler metrics $\{V_\alpha, g_\alpha^T\}$ is called a transverse almost Kähler metric, which we denote by g^T since they are isometric on the overlaps. We also write $\nabla^T, Rm^T, Ric^T, R^T$ for its Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature. It should be emphasized that, when restricted on \mathcal{D} , $\{d\eta/2, \Phi, g\}$ on U_α is isometric to $\{\omega_\alpha, \Phi_\alpha, g_\alpha^T\}$ via $d\pi_\alpha : \mathcal{D}_p \rightarrow T_{\pi_\alpha(p)}V_\alpha$. So we can also define $\{\mathcal{D}, d\eta/2, \Phi|_{\mathcal{D}}, g|_{\mathcal{D}}\}$ as an almost transverse Kähler structure via this isomorphism. This isomorphism will play an important role in our computations in the following sections. The transverse scalar curvature R^T also lifts to M as a global function. When (M, η, ξ, Φ, g) is Sasaki, it is easy to see that Φ_α is a complex structure on V_α , hence it defines the *transverse complex structure* or *transverse holomorphic structure* on the Reeb foliation. In particular, we can choose coordinates charts $V_\alpha \subset \mathbb{C}^n$ with coordinates (z^1, \dots, z^n) such that

$$\Phi_\alpha \frac{\partial}{\partial z_i} = \sqrt{-1} \frac{\partial}{\partial \bar{z}_i};$$

this with ω_α gives its transverse Kähler structure (for example see [16] Section 3). The corresponding charts U_α with coordinates (x^0, z^1, \dots, z^n) are called foliations charts.

Now let (M, η, ξ, Φ) be a compact Sasaki manifold.

Definition 2.2. A p -form θ on M is called basic if

$$\iota_\xi \theta = 0, L_\xi \theta = 0.$$

Let Λ_B^p be the sheaf of germs of basic p -forms and $\Omega_B^p = \Gamma(S, \Lambda_B^p)$ the set of sections of Λ_B^p .

The exterior differential preserves basic forms. We set $d_B = d|_{\Omega_B^p}$. There is a natural splitting of $\Lambda_B^p \otimes \mathbb{C}$ such that

$$\Lambda_B^p \otimes \mathbb{C} = \oplus \Lambda_B^{i,j},$$

where $\Lambda_B^{i,j}$ is the bundle of type (i, j) basic forms. We thus have the well defined operators (for example, see Section 4 [16])

$$\begin{aligned} \partial_B : \Omega_B^{i,j} &\rightarrow \Omega_B^{i+1,j}, \\ \bar{\partial}_B : \Omega_B^{i,j} &\rightarrow \Omega_B^{i,j+1}. \end{aligned}$$

Then we have $d_B = \partial_B + \bar{\partial}_B$. Set $d_B^c = \frac{1}{2}\sqrt{-1}(\bar{\partial}_B - \partial_B)$. It is clear that

$$d_B d_B^c = \sqrt{-1} \partial_B \bar{\partial}_B, d_B^2 = (d_B^c)^2 = 0.$$

Using the foliation chart, we can also check that

$$d_B^c = \frac{1}{2}\sqrt{-1}^{i-j} \Phi \circ d_B \text{ on } \Omega_B^{i,j}.$$

As in [18, 19], we consider the space of Sasaki metrics as follows

$$\mathcal{H} = \{\phi \in C_B^\infty(M), \eta_\phi \wedge (d\eta_\phi)^n \neq 0, \eta_\phi = \eta + d_B^c \phi\},$$

which can be viewed as an analogue of the space of Kähler metrics in a fixed Kähler class studied in [23, 29, 9]. For any $\phi \in \mathcal{H}$, we can define a new Sasaki metric $(\eta_\phi, \xi, \Phi_\phi, g_\phi)$ with the same Reeb vector field ξ such that

$$\eta_\phi = \eta + d_B^c \phi, \Phi_\phi = \Phi - \xi \otimes d_B^c \phi \circ \Phi.$$

Note that \mathcal{D} varies and the K-contact structure Φ_ϕ varies when ϕ varies but $(\eta_\phi, \xi, \Phi_\phi, g_\phi)$ has the same transverse holomorphic structure as (η, ξ, Φ, g) (Prop. 4.1 in [16], [1]); for example, we have the relation when ϕ varies,

$$\Phi \circ d_B = \Phi_\phi \circ d_B.$$

On the contrary, if $(\tilde{\eta}, \xi, \tilde{\Phi}, \tilde{g})$ is another Sasaki structure with the same Reeb vector field ξ and the same transverse Kähler structure, then there exists a unique function $\phi \in \mathcal{H}$ up to addition of a constant (e.g. [13]) such that

$$d\tilde{\eta} = d\eta_\phi = d\eta + \sqrt{-1}\partial\bar{\partial}\phi.$$

We shall denote this set by $\mathcal{S}(\xi, \bar{J})$ as in [2], where \bar{J} denotes a fixed transverse homomorphic structure on $\nu(\mathcal{F}_\xi)$. Note that $\tilde{\eta}$ does not have to be η_ϕ , but the transverse geometry of \tilde{g} is determined by $d\tilde{\eta}, \bar{J}$. Hence $\mathcal{S}(\xi, \bar{J})$ can be viewed as the analogue of the set of Kähler metrics in a fixed Kähler class. Boyer-Galicki-Simanca [2, 3] proposed to seek the extremal Sasaki metrics to represent $\mathcal{S}(\xi, \bar{J})$, by extending Calabi's extremal problem to Sasaki geometry (see [16] also). A metric is called a *transverse extremal Kähler metric*. To seek an extremal metric in $\mathcal{S}(\xi, \bar{J})$ is then reduced to find a function $\phi \in \mathcal{H}$ such that the Sasaki metric $(\eta_\phi, \xi, \Phi_\phi, g_\phi)$ is transverse extremal.

We conclude this section with the following proposition, which will be used in the following sections. Let (M, g) be a Sasaki metric and g^T be its transverse Kähler metric. Let $U_\alpha, (x^0, z^1, \dots, z^n)$ be a foliation chart and $\pi_\alpha : U_\alpha \rightarrow V_\alpha$ such that

$$d\pi_\alpha : D_p \rightarrow T_{\pi_\alpha(p)} V_\alpha.$$

Let

$$g_{i\bar{j}}^T = g^T \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right)$$

and $g_T^{i\bar{j}}$ be its inverse. It is clear that, for a basic function ϕ ,

$$\Delta^T \phi = g_T^{i\bar{j}} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}$$

lifts to M hence defines a function on M . Let ϕ, ψ be basic functions, then $d\pi_\alpha(\nabla \phi) = \nabla^T \phi$ and we have

$$(d_B \phi, d_B \psi)_g = (\nabla \phi, \nabla \psi)_g = (\nabla^T \phi, \nabla^T \psi)_{g^T}.$$

It is also clear that for a basic function ϕ ,

$$\Delta^T \phi = \Delta \phi.$$

3. THE SPACE OF SASAKI METRICS AS A SYMMETRIC SPACE

In this section we show that \mathcal{H} is an infinite-dimensional symmetric space, as in Kähler setting [23, 29, 9]. It can be viewed as the negatively curved dual space of \mathcal{G} , see [9] for the Kähler setting.

Let (M, ξ, η, Φ) be a compact Sasaki manifold. We shall briefly recall the geometric structure on \mathcal{H} introduced in [18]. For any $\phi \in \mathcal{H}$, we can define a metric on $T_\phi \mathcal{H}$

$$\langle \psi_1, \psi_2 \rangle_\phi = \int_M \psi_1 \psi_2 d\mu_\phi, \forall \psi_1, \psi_2 \in T\mathcal{H},$$

where $d\mu_\phi = (2^n n!)^{-1} \eta_\phi \wedge (d\eta_\phi)^n$ is the volume form determined by g_ϕ . If $\phi(t) : [0, 1] \rightarrow \mathcal{H}$ is a path, the geodesic equation can be written as

$$\ddot{\phi} - \frac{1}{4} |d_B \dot{\phi}|_\phi^2 = 0.$$

Thus $\psi(t)$ is a field of tangent vectors along the path $\phi(t) \in \mathcal{H}$, the covariant derivative along the path is given by

$$D_\phi \psi = \frac{d}{dt} \psi - \frac{1}{4} (d_B \dot{\phi}, d_B \psi)_\phi.$$

Guan-Zhang proved [18] that this connection is torsion free and compatible with the metric

$$\frac{d}{dt} \|\psi\|_\phi^2 = 2 \langle D_t \psi, \psi \rangle;$$

They also proved that the corresponding sectional curvature of \mathcal{H} is nonpositive and there is a Riemannian decomposition

$$\mathcal{H} = \mathcal{H}_0 \times \mathbb{R},$$

where $\mathcal{H}_0 = \mathcal{H}/\mathbb{R}$.

Theorem 3.1. *\mathcal{H} is an infinite dimensional symmetric space; the curvature of the connection D is covariant constant. At a point $\phi \in \mathcal{H}$ the curvature is given by*

$$R_\phi(\psi_1, \psi_2)\psi_3 = -\frac{1}{16} \{ \{ \psi_1, \psi_2 \}_\phi, \psi_3 \}_\phi,$$

where $\{ , \}_\phi$ is the Poisson bracket on $C_B^\infty(M)$ induced by the contact form η_ϕ .

Proof. First we observe that (M, η_ϕ) are all equivalent as contact structures. Let $\phi(t)$ be a path starting at 0 in \mathcal{H} and consider the t dependent vector field

$$X_t = -\frac{1}{4} \nabla_{g(t)} \dot{\phi},$$

where $g(t)$ is the metric determined by $(M, \xi, \eta(t), \Phi(t))$. Let $f_t : M \rightarrow M$ be the 1-parameter family of diffeomorphisms obtained by integrating $X(t)$ with $f_0 = 1_M$. Then we compute that

$$\frac{d}{dt} (f_t^* \eta(t)) = f_t^* \left(L_{X_t} \eta(t) + \frac{d}{dt} \eta(t) \right)$$

For any t , $d\phi(t)(\xi) = 0$, we get that $d\dot{\phi}(\xi) = 0$. Hence $\langle \xi, \nabla_{g(t)} \dot{\phi} \rangle_{g(t)} = 0$. This implies that $X_t \in \ker(\eta(t))$. So we can compute

$$L_{X_t} \eta(t) = \iota_{X_t} d\eta(t) + d(\iota_{X_t} \eta(t)) = \iota_{X_t} d\eta(t).$$

For each t fixed and $Y \in TM$,

$$\begin{aligned} \iota_{X_t} d\eta(t)(Y) &= d\eta(t)(X_t, Y) = -2 \langle X_t, \Phi(t)Y \rangle_{g(t)} = \frac{1}{2} \langle \nabla \dot{\phi}, \Phi(t)Y \rangle \\ &= \frac{1}{2} d_B \dot{\phi}(\Phi(t)Y) = -\frac{1}{2} (\Phi(t) \circ d_B) \dot{\phi}(Y) = -d_B^c \dot{\phi}(Y). \end{aligned}$$

On the other hand, the t -derivative of $\eta(t)$ is $d_B^c \dot{\phi}$. So we have

$$\frac{d}{dt} f_t^*(\eta(t)) = f_t^* \left(L_{X_t} \eta(t) + \frac{d}{dt} \eta(t) \right) = 0,$$

i.e. the diffeomorphism f_t gives the desired contactomorphism from (M, η_0) to $(M, \eta(t))$. Now let $\mathcal{Y} \subset \mathcal{H} \times \text{Diff}(M)$ be the set of pairs (ϕ, f) such that $f^* \eta_\phi = \eta$. This is a principal bundle over \mathcal{H} with structure group \mathcal{G} . Then the discussion above shows that the connection D on the tangent space of \mathcal{H} is induced from a \mathcal{G} connection on $\mathcal{Y} \rightarrow \mathcal{H}$ via the action of \mathcal{G} on $C_B^\infty(M)$; that is, we have a connection preserving bundle isomorphism

$$T\mathcal{H} = \mathcal{Y} \times_{\mathcal{G}} C_B^\infty(M).$$

And the connection D is compatible with the metric since the L^2 norm on $C_B^\infty(M)$ is G invariant. Now we compute the curvature tensor of \mathcal{H} . To do this we consider a 2-parameter family $\phi(s, t)$ in \mathcal{H} , and a vector field $\psi(s, t)$ along $\phi(s, t)$. We denote s and t derivatives by suffixes ϕ_s, ϕ_t etc. The curvature is given by

$$R_\phi(\phi_s, \phi_t)\psi = (D_s D_t - D_t D_s)\psi.$$

It is clear that $R(\phi_s, \phi_t)\psi$ is linear in ϕ, ψ . It is also clear that all functions involved are basic functions. Hence we can do computation by using the transverse Kähler structure defined by $\{\mathcal{D}_\phi, \eta_\phi, \Phi_\phi, g_\phi\}$. We can write, for example, in a foliation chart $U_\alpha \subset M$,

$$(d_B \psi_1, d_B \psi_2)_\phi = (\nabla^T \psi_1, \nabla^T \psi_2)_{g^T},$$

where ∇^T is the Levi-Civita connection of the transverse Kähler metric g^T defined by $\{\mathcal{D}_\phi, \eta_\phi, \Phi_\phi, g_\phi\}$. We also consider the projection $\pi_\alpha : U_\alpha \rightarrow V_\alpha$ and the corresponding complex structure on V_α is denoted as Φ_ϕ^α .

Expanding out, we compute

(3.1)

$$\begin{aligned} 4R_\phi(\phi_s, \phi_t)\psi = & \left(\frac{\partial}{\partial t} (\nabla^T \psi, \nabla^T \phi_s)_{g^T} - \frac{\partial}{\partial s} (\nabla^T \psi, \nabla^T \phi_t)_{g^T} \right) \\ & + ((\nabla^T \psi_s, \nabla^T \phi_t)_{g^T} - (\nabla^T \psi_t, \nabla^T \phi_s)_{g^T}) \\ & + (\nabla^T (\nabla^T \psi, \nabla^T \phi_s)_{g^T}, \nabla^T \phi_t)_{g^T} - (\nabla^T (\nabla^T \psi, \nabla^T \phi_t)_{g^T}, \nabla^T \phi_s)_{g^T} \end{aligned}$$

On the other hand, we can write the Possion bracket $\{, \}$ as follows. For any $f, h \in C_B^\infty(M)$,

$$\{f, h\}_\phi = \eta_\phi([X_f, X_h]) = -d\eta_\phi(X_f, X_h) = 2(\nabla^T f, \Phi_\phi^\alpha \nabla^T h)_{g^T}.$$

Hence we compute

$$(3.2) \quad \frac{1}{4} \{ \{ \phi_s, \phi_t \}, \psi \}_\phi = (\nabla^T (\nabla^T \phi_s, \Phi_\phi^\alpha \nabla^T \phi_t)_{g^T}, \Phi_\phi^\alpha \nabla^T \psi)_{g^T}.$$

Now we claim that the right hand side of (3.1) coincides with the right hand side of (3.2). Note that all the quantities above are only involved with the transverse Kähler structure and we can express all these quantities in terms of the Kahler metric g_α^T on V_α . Hence this reduces the computations to the Kahler setting and the same argument in [9] (Page 18) can be carried over here directly without any change. The only difference is that the Kähler form of g_α^T is $d\eta_\phi/2$ while the Possion

bracket is defined by η_ϕ and $d\eta_\phi$. So there is difference of a constant factor 4 and we get

$$R_\phi(\phi_s, \phi_t)\psi = -\frac{1}{16}\{\{\phi_s, \phi_t\}_\phi, \psi\}_\phi.$$

The expression of the curvature tensor in terms of Poisson brackets shows that R_ϕ is invariant under the action of the group \mathcal{G} ; it follows that R_ϕ is covariant constant and hence \mathcal{H} is indeed an infinite-dimensional symmetric space. \square

Remark 3.2. *It is clear that the Riemannian decomposition $\mathcal{H} = \mathcal{H}_0 \times \mathbb{R}$ corresponds to the Lie algebra decomposition $C_B^\infty(M) = C_{B,0}^\infty(M) \oplus \mathbb{R}$. When (M, ξ, η) is quasiregular, then ξ generates one parameter subgroup of \mathcal{G} which lies in the center of \mathcal{G} (in quasiregular case, the orbits of ξ are compact, hence are all circles, then ξ generates a circle action on M which preserves η), then we have the group isomorphism*

$$\mathcal{G} \approx \mathcal{G}/S^1 \times S^1,$$

which corresponds the Lie algebra decomposition $C_B^\infty(M) = C_{B,0}^\infty(M) \oplus \mathbb{R}$.

4. TRANSVERSE SCALAR CURVATURE AS A MOMENT MAP

In this section we show that the transverse scalar curvature of a Sasaki metric is a moment map with respect to the strict contactomorphism group \mathcal{G} . When (V, ω) is a symplectic manifold and let \mathcal{J} be the space of almost complex structures which are compatible with ω . Then scalar curvature $R : \mathcal{J} \rightarrow C^\infty(M)$ is a moment map of symplectomorphism group of (V, ω) which acts on \mathcal{J} [14, 9]. This point of view of moment map can be carried over to Sasaki geometry with only slight modification.

First let us recall the definition of a moment map. Let (V, ω) be a symplectic manifold with a symplectic form ω . Suppose that a Lie group G acts on V via symplectomorphisms. Let \mathfrak{g} be the Lie algebra of G . Then any $\zeta \in \mathfrak{g}$ induces a one-parameter subgroup $\{\phi(t)\}$ in G , and $\{\phi(t)\}$ induces a vector field X_ζ on V since G acts on V . A moment map for the G -action on (V, ω) is a map $\mu : V \rightarrow \mathfrak{g}^*$ such that

$$d\langle \mu, \zeta \rangle = \iota_{X_\zeta} \omega.$$

Here $\langle \mu, \zeta \rangle$ is the function from V to \mathbb{R} defined by $\langle \mu, \zeta \rangle(x) = \langle \mu(x), \zeta \rangle$. We also require that the moment map μ is G -equivariant with respect to the co-adjoint action of G on \mathfrak{g}^* .

Consider a compact contact manifold (M^{2n+1}, η) . Let \mathcal{K} be the space of K -contact-complex structures on M which are compatible with η ; we shall assume \mathcal{K} is not empty. We shall show that the space \mathcal{K} can be endowed with the structure of an infinite-dimensional Kähler manifold. Let \mathcal{G} be the strict contactomorphism group which preserves the contact form η . Then \mathcal{G} acts on \mathcal{K} via

$$(f, \Phi) \rightarrow f_*^{-1} \Phi f_*.$$

For simplicity we consider $\Phi \in \mathcal{K}$ is integrable. Let $\mathcal{K}_{int} \subset \mathcal{K}$ defined as

$$\mathcal{K}_{int} = \{\Phi \in \mathcal{K} : \Phi \text{ is integrable}\}.$$

We assume \mathcal{K}_{int} is not empty. We want to identify a moment map for the action of \mathcal{G} on \mathcal{K}_{int} . For each $\Phi \in \mathcal{K}_{int}$, (M, η, Φ) defines a Sasaki metric g , and let $\{\mathcal{D}, \frac{1}{2}d\eta, \Phi|_{\mathcal{D}}\}$ be its transverse Kähler metric. We have

Theorem 4.1. *The map $\Phi \rightarrow R^T(\Phi)$ is an equivariant moment map for the \mathcal{G} -action on \mathcal{K}_{int} , under the natural pairing:*

$$(R^T, H) \rightarrow \int_M R^T H d\mu_\eta.$$

First observe that

$$T_K \mathcal{K} = \{A : TM \rightarrow TM; A\xi = 0, L_\xi A = 0, A\Phi + \Phi A = 0, d\eta(\Phi A \cdot, \cdot) + d\eta(\cdot, \Phi A) = 0\}.$$

Note that if $A \in T_K \mathcal{K}$, then $\Phi A \in T_K \mathcal{K}$; hence we can define a natural almost structure $J_K : T_K \mathcal{K} \rightarrow T_K \mathcal{K}$ on \mathcal{K} such that

$$J_K A = \Phi A.$$

We can define a natural metric on \mathcal{K} . For any $A \in T_K \mathcal{K}$, we can identify A with

$$A(X, Y) = g_\Phi(AX, Y), X, Y \in TM.$$

It is clear that A is anti- Φ invariant and symmetric. We can identify A with a section of $T^*M \otimes T^*M$. The metric g_Φ induces a metric on $T^*M \otimes T^*M$. So we can define a metric on $T_\Phi \mathcal{K}$ by integration over M ,

$$g_K(A, B) = \int_M \langle A, B \rangle_{g_\Phi} d\mu.$$

For $p \in M$, if $\mathcal{D}_p = \text{span}\{e_i : 1 \leq i \leq 2n\}$, and $A_p : \mathcal{D}_p \rightarrow \mathcal{D}_p$ such that $A_p e_i = A_i^j e_j$. Then $\langle A, B \rangle_p = \text{trace}(A_p B_p) = A_i^j B_j^i$. It is clear that g_K is compatible with J_K ; hence it defines an Hermitian metric on \mathcal{K} . As in the (almost) Kähler setting, (\mathcal{K}, g_K, J_K) induces a natural Kähler structure on \mathcal{K} .

Proposition 4.2. *(\mathcal{K}, g_K, J_K) is Kähler and \mathcal{K}_{int} is an analytic submanifold.*

Note that $\Phi \in \mathcal{K}$ can be identified as an almost complex structure on \mathcal{D} which is invariant under the action generated by ξ , hence \mathcal{K} can be viewed the space of sections of a fibre bundle on M with fibre the complex homogeneous space $Sp(2n, \mathbb{R})/U(n)$. Then \mathcal{K} inherits a complex structure from that of the fibre. Indeed, there is a global holomorphic coordinate chart if we use the ball model of the Siegel upper half space in the usual way, see [14] Theorem 4.2 and [9] for more details.

Now we are in the position to prove Theorem 4.1.

Proof. Fix a K-contact-complex structure $\Phi \in \mathcal{K}_{int}$; let

$$\mathcal{P} : C_B^\infty(M) \rightarrow \Gamma(T\mathcal{K})$$

be the operator representing the infinitesimal action of \mathcal{G} on \mathcal{K}_{int} and

$$\mathcal{Q} : \Gamma(T\mathcal{K}) \rightarrow C_B^\infty(M)$$

be the operator which represents the derivative of $\Phi \rightarrow R^T(\Phi)$. By the definition of a moment map we need to show that for all $\phi \in C_B^\infty(M)$, $A \in \Gamma(T\mathcal{K})$,

$$g_K(J_K \mathcal{P}(\phi), A) = \langle \phi, \mathcal{Q}(A) \rangle.$$

The operator \mathcal{P} can be factored as $\mathcal{P} = \mathcal{P}_2 \mathcal{P}_1$, where \mathcal{P}_1 maps ϕ to the Hamiltonian vector field X_ϕ and \mathcal{P}_2 maps a vector field X to the infinitesimal variation given by

the Lie derivative $L_X \Phi$. We choose a coordinate chart $U_\alpha = \{(x^0, x^1, \dots, x^{2n})\}$ as in Section 2, such that

$$\eta = dx^0 - a_i dx^i, \xi = \frac{\partial}{\partial x_0}, \mathcal{D} = \text{span} \left\{ \frac{\partial}{\partial x_i} + a_i \frac{\partial}{\partial x^0}, 1 \leq i \leq 2n \right\}.$$

Let $V_\alpha = \{(x^1, \dots, x^{2n})\}$ such that $\pi_\alpha : U_\alpha \rightarrow V_\alpha$ is a submersion. Recall $\omega_\alpha = d\eta/2 = (\omega_{ij}), \Phi_\alpha = (\Phi_i^j)$ and g_α^T the transverse Kähler structure. For simplicity we suppress α and write $g_\alpha^T = (g_{ij}) = (\omega_{ik} \Phi_j^k)$, $1 \leq i, j \leq 2n$. It is clear that on U_α , we can write $A = (A_i^j)$ for some functions A_i^j independent of x^0 , as we have discussed in Section 2. Hence we can define A_α on V_α , which we still denote as $A_\alpha = (A_i^j)$. Note that all quantities involved in the computations can be written locally on V_α and we have the isometry of $\{\mathcal{D}, \frac{1}{2}d\eta, \Phi|_{\mathcal{D}}\}$ with g^T . So we can do all (local) computations on V_α . Since $\iota_{X_\phi} d\eta = -d\phi$, we have

$$\omega \left(X_\phi, \frac{\partial}{\partial x_j} \right) = -\frac{1}{2} \frac{\partial \phi}{\partial x_j}.$$

It follows that

$$X_\phi = X^i \frac{\partial}{\partial x_i} = -\frac{1}{2} \Phi_k^j g^{ki} \frac{\partial \phi}{\partial x_j}.$$

Suppose that

$$L_{X_\phi} \Phi = B_j^i \frac{\partial}{\partial x_i} \otimes dx_j;$$

then we have

$$B_j^i \frac{\partial}{\partial x_i} = L_{X_\phi} \Phi \left(\frac{\partial}{\partial x_j} \right) = L_{X_\phi} \left(\Phi \frac{\partial}{\partial x_j} \right) - \Phi \left(L_{X_\phi} \frac{\partial}{\partial x_j} \right).$$

Hence

$$B_j^i = \frac{1}{2} \Phi_j^k \Phi_l^p g^{li} \phi_{kp} - \frac{1}{2} \Phi_k^i g^{kl} \Phi_l^p \phi_{pj}.$$

It follows that

$$\mathcal{P}(\phi) = \frac{1}{2} (\Phi_i^k \Phi_j^l \phi_{kl} - \phi_{ij}).$$

We can then compute the pairing

$$g_{\mathcal{K}}(J_{\mathcal{K}} \mathcal{P}(\phi), A) = - \int_M \langle \mathcal{P}(\phi), J_{\mathcal{K}} A \rangle d\mu = \int_M g^{jk} g^{pl} \Phi_p^i \phi_{ij} A_{kl} d\mu,$$

where $A_{ij} = g_{ik} A_j^k$, $1 \leq i, j, k, l, p \leq 2n$. Now we compute $\mathcal{Q}(A)$. Since $(\delta g^T)_{ij} = \omega_{ik} A_j^k = \Phi_i^k A_{kj}$, the well-known formula of the variation of the scalar curvature R^T is given by

$$\delta R^T = -2g^{ij} g^{kl} (\delta g)_{il} R_{jk}^T + g^{ik} g^{jl} (\delta g^T)_{ij,kl} - g^{ij} g^{kl} (\delta g^T)_{ij,kl}.$$

Note that the metric g^T and the Ricci curvature R_{ij}^T are Φ -invariant and A is anti Φ -invariant, we can get that

$$g^{ij} g^{kl} (\delta g^T)_{il} R_{jk}^T = 0, g^{ij} (\delta g^T)_{ij} = 0.$$

It follows that

$$\mathcal{Q}(A) = \delta R^T = g^{ik} g^{jl} (\delta g^T)_{ij,kl} = g^{ik} g^{jl} (\Phi_i^p A_{pj})_{,kl}.$$

We can compute, integration by parts,

$$\langle \phi, \mathcal{Q}(A) \rangle = \int_M \phi g^{ik} g^{jl} (\Phi_i^p A_{pj})_{,kl} d\mu = \int_M g^{ik} g^{jl} \Phi_i^p \phi_{kl} A_{pj} d\mu.$$

The integration by parts above can be justified as in Proposition 4.5 [20], by noting that all the integrands involved are basic, namely, are invariant under the group action generated by ξ . Hence we get the desired equality

$$g_K(J_K \mathcal{P}(\phi), A) = \langle \phi, \mathcal{Q}(A) \rangle.$$

□

Remark 4.3. *The similar results should hold for \mathcal{K} , the space of K -contact complex structures which are compatible with a fixed contact form. One should consider the transverse Hermitian scalar curvature defined by the almost transverse Kähler structure, but have to take the Nijenhuis tensor into account, as in [9] for the almost Kähler setting. We shall assume integrability for simplicity.*

5. CANONICAL METRICS IN SASAKI GEOMETRY

Calabi's extremal metric problem [4, 5] in Kähler geometry are closely related to the geometry of the space of Kähler metrics and the space of complex structures which are compatible with a fixed Kähler form. We refer the reader to the papers of S.K. Donaldson [8, 9] for details. Calabi's extremal problem can also be extended to Sasaki geometry, see [2, 3] for example.

We have seen that the standard picture in [8, 9] can be applied to Sasaki geometry; hence canonical metric problems in Sasaki geometry are also closely related to the geometry of \mathcal{H} and \mathcal{K} . We shall roughly repeat the picture described in [8, 9] for Sasaki context as follows.

If (V, ω, J) is a Kähler manifold and assume there is an action of a compact connected group G on V which preserves the Kähler structure. Let μ be the corresponding moment map. This induces a holomorphic action of the complexified group $G^{\mathbb{C}}$. Then the Kempf-Ness theorem relates the complex quotient by $G^{\mathbb{C}}$ to the symplectic reduction by G ([12]).

Now let (M, η, ξ) be a Sasaki manifold. The strict contactomorphism group \mathcal{G} acts on the Kähler manifold \mathcal{K}^{int} as holomorphic isometries, and the transverse scalar curvature $R^T : \mathcal{K}^{int} \rightarrow C_B^{\infty}(M)$ is an equivariant moment map of \mathcal{G} -action on \mathcal{K}^{int} . Let $m = R^T - \underline{R}$, where \underline{R} is the average of the transverse scalar curvature, depending only on the basic class $[d\eta]_B$. Then $m : \mathcal{K}^{int} \rightarrow C_B^{\infty}(M)$ is also a moment map of \mathcal{G} . If there is a *complexification* group \mathcal{G}^c of \mathcal{G} , then by the *standard picture*, a constant transverse scalar curvature metric, which is a zero point of the moment map m , corresponds to a stable complex orbit of \mathcal{G}^c action and we expect the identification

$$K_s^{int}/\mathcal{G}^c = m^{-1}(0)/\mathcal{G}.$$

In general, \mathcal{G}^c might not exist. However we can complexify the Lie algebra of \mathcal{G} and it acts on \mathcal{K}^{int} automatically since \mathcal{K}^{int} is a complex manifold. At each point $\Phi \in \mathcal{K}^{int}$ we get a subspace of $T_{\Phi} \mathcal{K}^{int}$ spanned by this complexification action and these subspaces form an integrable, holomorphic distribution on \mathcal{K}^{int} . Thus we get a distribution of \mathcal{K}^{int} which plays the role of the complex orbits. By definition, the infinitesimal action of ϕ on \mathcal{K}^{int} is given by $L_{X_{\phi}} \Phi$ for any $\Phi \in T_{\Phi} \mathcal{K}^{int}$, then the infinitesimal action of a function $\sqrt{-1}\phi \in C_B^{\infty}(M) \otimes \mathbb{C}$ is $L_{X_{\phi}} \Phi$, the natural

action of ΦX_ϕ on Φ . Thus the geometric effect of applying $\sqrt{-1}\phi$ is the same as keeping the transverse holomorphic structure induced by Φ fixed and varying the transverse symplectic form $\frac{1}{2}d\eta$ to

$$\frac{1}{2}d\tilde{\eta} = \frac{1}{2}d\eta - L_{\Phi X_\phi} \left(\frac{1}{2}d\eta \right) = \frac{1}{2}d\eta + \sqrt{-1}\partial\bar{\partial}\phi,$$

which corresponds to the space of Sasaki metrics \mathcal{H} .

We shall recall the definition of K-energy in Sasaki geometry [22, 19], which is defined on \mathcal{H} by specifying its variation

$$\delta\mathcal{M} = - \int_M \delta\phi(R^T - \underline{R})d\mu_\phi.$$

A critical point of \mathcal{M} is a constant transverse scalar curvature and \mathcal{M} is convex along geodesics in \mathcal{H} [19].

We can also ask the similar questions as in [9] regarding the existence of constant transverse scalar curvature metrics, which can be viewed as an analogue of the Hilbert criterion for stability in geometric invariant theory.

Problem 5.1. *The following are equivalent:*

- (1). *There is no critical Kähler metric in \mathcal{H}_0 .*
- (2). *There is an infinite geodesic ray $\phi_t \in \mathcal{H}_0$, such that $t \rightarrow \infty$, the derivative of K-energy is less than zero along the geodesic ray ϕ_t ,*

$$\int_M \dot{\phi}(\underline{R} - R^T)d\mu_\phi < 0.$$

- (3). *For any point $\phi \in \mathcal{H}_0$ there is a geodesic ray as in (2) starting at ϕ_0 .*

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